Practical Optimisation of Noise and Distortion in Sallen and Key Filter Sections

PETER J. BILLAM

Filtek A.G., Biel, Switzerland

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This communication presents simple analytic formulas for the noise and distortion aggravation performance of Sallen and Key filter sections, under the approximation that the resistor noise contribution is small compared with the amplifier noise. This performance is optimised analytically and subject to component constraints.

1 NOISE

Noise in active filter has two main origins: the network resistors and the amplifier. The analytical optimisation used in this communication is made possible because in practical circuits, the resistor noise contribution is small compared with the amplifier noise contribution ([1], sec. VI). Thus the noise minimisation problem reduces to the minimisation of the amplifier-to-output transfer function (not to mention the choice of a quiet amplifier!). After this optimisation, if desired, the resistor noise contribution can also be minimised by a simple denormalisation of resistor and capacitor values; indeed the designer who is interested in the ultimate noise figure can always eliminate the resistor noise contribution, since by either indulging in large capacitors, or by bufferring the amplifier output to permit low R_{min}, he can make R as small as he likes.

2 DISTORTION

A recent paper by the author [2] has shown that the fact of employing an amplifier in a filter section worsens the amplifier's distortion products by an amount, the Distortion Aggravation Factor, or F_{DAG} equal to the value of the transfer function from amplifier to filter-output.

F_{DAG} is frequently worse at the centre-frequency $\omega = 1/RC = \omega_0$, and in practical circuits $F_{DAG}(\omega = \omega_0)$ may be of the order of between 1 and 10^3.

Thus minimising F_{DAG} optimises both noise and distortion simultaneously, a most fortunate and unusual situation in electronics. Of course the reduction of resistor noise by denormalisation will maximise distortion, because of the increased amplifier loading, a more normal state of affairs.

3 INPUT-REFERRED OR OUTPUT-REFERRED NOISE ?

Depending on the design context in which the filter occurs, the designer may be interested in optimising either the output-referred noise (V_{out}^{noise}) or the input-referred noise (V_{in}^{noise}/g).

For example, if the filter stage occurs as part of an amplifying chain, the input-referred noise should be optimised since the gain of the filter can then be used as part of the gain of the amplifying stages. On the other hand, if the filter is to be used to treat an already full-level signal, the output-noise should be optimised, since any gain "g" in the filter will have to be cancelled out by an attenuator of gain "$1/g$" fitted in front of the filter to avoid the filter clipping in the passband.

There are traps here for the unwary optimiser. For example, in a multiamplifier stage, the noise can be "optimised" by giving the first amplifier a gain $g \to \infty$, and the last amplifier a gain of $g \to 0$, a trivial result. This trap was, for example responsible for the spectacular improvement in Haase' and Bruton's paper [1]. A better performance could have been obtained by fitting a high-gain preamplifier and following attenuator.

Distortion and input-referred noise are both optimised by minimising F_{DAG}, output-referred noise by minimising $g \cdot F_{DAG}$. Both these problems will be considered.

4 SALLEN AND KEY LOW-PASS SECTION

We consider the circuit (Fig. 1), whose transfer function is easily shown to be

$$V_{out}/V_{in} = g \ / \ (s^2 + s/Q + 1) \tag{1}$$

where

$$s = j \cdot \omega \cdot R \cdot C \quad \text{and} \quad Q = \gamma\xi \ / \ (\xi^2 + (1-g)\gamma^2 + 1) \tag{2}$$

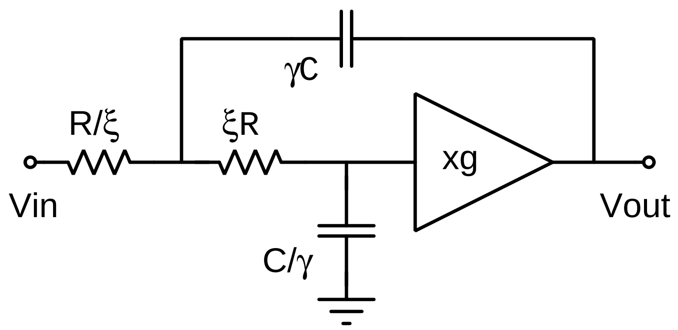

Figure 1: The Sallen and Key filter

4.1 Distortion Aggravation Factor

This being equal to the transfer function from amplifier to filter output, we consider the circuit in Fig. 2. It can be shown that

$$F_{DAG} = V_{out}/U_{in} = \frac{s^2 + s(1+\xi^2+\gamma^2)/\xi\gamma + 1}{s^2 + s(1+\xi^2+(1-g)\gamma^2)/\xi\gamma + 1} \tag{3}$$

Thus F_{DAG} is a function of frequency and reaches its maximum (i.e., worst) value at $s = j$, i.e., $\omega = 1/RC = \omega_0$

$$F_{DAG} = Q \cdot (\xi^2 + \gamma^2 + 1)/\xi\gamma \tag{4}$$

from (Eq. 2),

$$F_{DAG} = Q \cdot g \cdot \gamma/\xi + 1 \tag{5}$$

It will be seen shortly that this can be improved, while preserving the wanted filter transfer function, by increasing ξ and γ, i.e., by increasing the disparity between the two resistors and between the two capacitors.

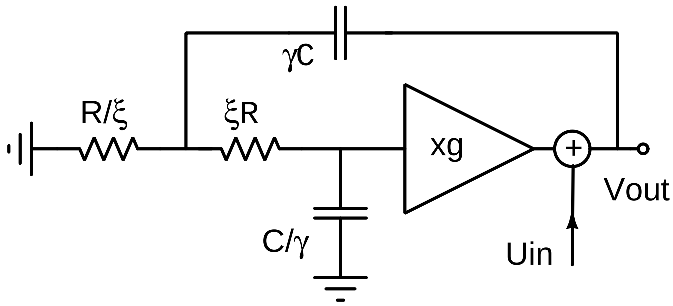

Figure 2: Calculation of F_{DAG}

5 COMPONENT CONSTRAINTS

We will assume that the usable range of component values is limited to between C_{min} and C_{max}, and R_{min} and R_{max}, say. Furthermore, the conditions on R and C interact since their product is fixed by the desired filter frequency:

$$C/\gamma > C_{min} \quad \text{and} \quad R/\xi > R_{min}$$

so multiplying

$$\xi\gamma < C \cdot R/(C_{min} \cdot R_{min}) \tag{6}$$

similarly,

$$\xi\gamma < C_{max} \cdot R_{max}/(C \cdot R) \tag{7}$$

Thus $\xi\gamma$ is bounded by the two conditions (6) and (7), and so our minimisations must be performed subject to an upper bound on $\xi\gamma$.

6 MINIMISING DISTORTION AND INPUT-REFERRED NOISE

We wish to minimise $F_{DAG} = Q \cdot g \cdot \gamma/\xi + 1$ from Eqn. 5.

Then since Q and $\xi\gamma$ remain constant, we must minimise $g \cdot \gamma^2$ to minimise F_{DAG}. Rearranging (2),

$$g\gamma^2 = 1 - \xi\gamma/Q + (\xi\gamma)^2/\gamma^2 + \gamma^2 \tag{8}$$

This is a minimum when

$$\delta(g\gamma^2)/\delta\gamma|_{Q,\,\xi\gamma\ constant} = 0 = -2(\xi\gamma)^2/\gamma^2 + 2\gamma$$

i.e., when

$$\xi = \gamma = \sqrt{\xi\gamma} \tag{9}$$

The practical procedure is then as follows:

1) Q is given and CR is known from $\pi f_0/2$
2) $\xi\gamma$ is chosen not to contavene either of the conditions (6) or (7)
3) ξ and γ are calculated from $\sqrt{\xi\gamma}$ (9)
4) g is calculated from (2)

For example, let us optimise the distortion and input-referred noise of a stage with $q = 5$, and the component constraint $\xi\gamma = 9.0$. Then (9) gives $\xi = \gamma = 3$, when (2) gives $g = 1.91111$, giving a resultant $F_{DAG} = Qg\gamma/\xi + 1 = 10.555$ (see Fig. 3). Whereas the "standard" design has $\xi = 1$, $g = 1$ and $\gamma = 10$, (see Fig 4.) and gives $F_{DAG} = Qg\gamma/\xi + 1 = 51.0$. The simplicity of these calculations compares favorably with that of a computer-based optimisation.

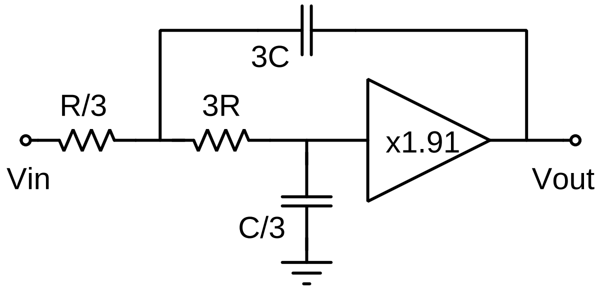

Figure 3: $Q = 5$, optimised for distortion and input noise.

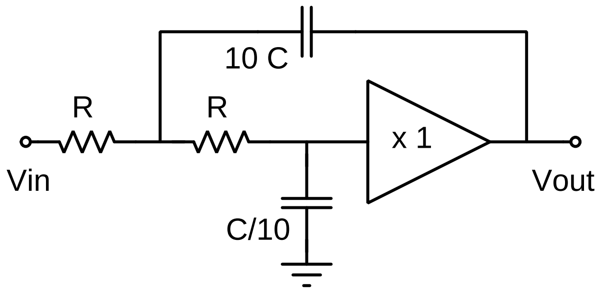

Figure 4: $Q = 5$, standard design.

6.1 Without Component Constraints

We should note that the most favorable optimisation, that is one without component constraints, i.e., $\xi\gamma \to \infty$ is, from (9), $\xi = \gamma \to \infty$; from (2), $g = 2 - 1/Q$ giving $F_{DAG}{}^{\omega=\omega0}_{(optimum)} = Qg\gamma/\xi + 1 = 2Q$ This result represents a useful "goodness factor" for evaluating the worth of the Sallen and Key circuit.

7 MINIMISATION OF OUTPUT-REFERRED NOISE

We wish to minimise

$$g \cdot F_{DAG}{}^{\omega=\omega0} = Q/\xi\gamma(1/\gamma^2 - \xi/Q\gamma + \xi^2/\gamma^2 + 1)(\xi^2 + \gamma^2 + 1)$$

from (2) and (5). We therefore set $\delta(g \cdot F_{DAG})/\delta\xi|_{Q,\xi\gamma\ constant}$ equal to 0, which reduces to

$$\gamma^4 + (2(\xi\gamma)^3/Q - 4(\xi\gamma)^2)/\gamma^2 - 3(\xi\gamma)^4/\gamma - 1(\xi\gamma)^2 + \xi\gamma/Q - 1 = 0 \qquad (10)$$

and if Q is known, and $\xi\gamma$ taken from the stricter condition of (6) or (7), then this can easily be solved for γ by successive approximation. Then ξ is calculated from $\xi\gamma/\gamma$, and g from (2), as in Section 6.

For example, let us optimise the output-referred noise of our $Q = 5$ stage with $\xi\gamma = 9$. Equation (1) becomes

$$\gamma^4 + (291.6 - 324)/\gamma^2 - 19683/\gamma^4 - 161.2 = 0$$

which is satisfied by $\gamma = 9\,/\,3.9521 = 2.27727$, when $g = 1.2808$, giving a resultant $g \cdot F_{DAG} = g\,(12.114) = 15.516$. (See Fig. 5)

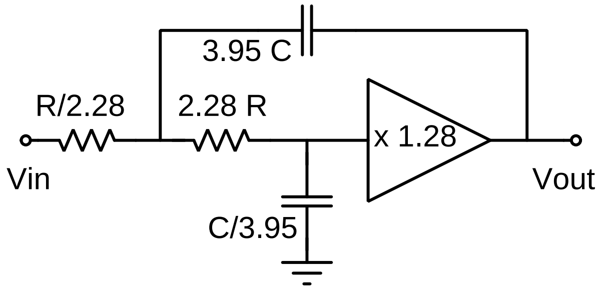

Figure 5: $Q = 5$, optimised for output-referred noise.

8 GAIN CONSTRAINT

It may also arise that the passband gain g of the filter is fixed by the application. In this case, no optimisation is possible except to choose $\xi\gamma$ as high as possible, and solve (2) for ξ and γ; i.e.,

for $g \neq 1$,

$$\gamma^2 = \frac{\xi\gamma/Q - 1 + \sqrt{(\xi\gamma/Q)^2 + 4(g-1)\xi^2\gamma^2}}{2(1-g)} \tag{11}$$

and for $g = 1$,

$$\gamma = \frac{\xi\gamma}{\sqrt{\xi\gamma/Q - 1}} \tag{12}$$

(See Fig. 6)

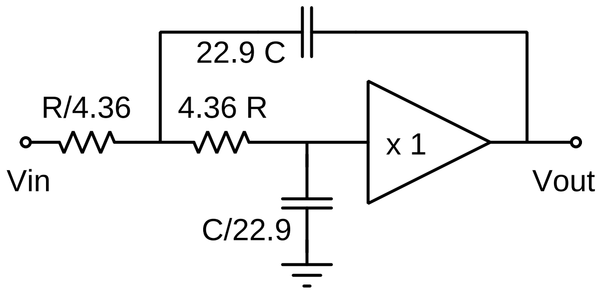

Figure 6: $Q = 5$, optimised for noise and distortion with $g = 1$

8.1 Without Component Constraints

We note that in the case where the gain is fixed, but the component constraints are relaxed, i.e. $\xi\gamma \to \infty$, then Eqn. 11 becomes, for $g \neq 1$,

$$\gamma^2 \to \xi\gamma \cdot \frac{1 + \sqrt{1 + 4q^2(g-1)}}{2Q(1-g)} \tag{13}$$

and Eqn. 12 becomes, for $g = 1$,

$$\gamma \to \sqrt{Q}\sqrt{\xi\gamma}$$
$$\xi \to \gamma/Q \tag{14}$$

giving

$$F_{DAG}^{\omega=\omega 0} = Q^2 + 1 \tag{15}$$

This should be compared with the distortion aggravation factor of $2Q$ that can be obtained by optimising g also (Section 7). Unity gain amplifiers should therefore be avoided unless the desired Q is approximately $Q \simeq 1$. They are also, of course, undesirable for sensitivity reasons [4].

9 AMPLIFIER LOADING

We state here for completeness that the maximum current the Sallen and Key network can draw from its amplifier occurs in the stopband, and is equal to $\xi V_{in}/R$.

10 CONCLUSIONS

The Sallen and Key second-order filter section has been optimised analytically for distortion, for input-referred noise, and for out-referred noise, under the realistic assumtion that the resistor noise is small compared with the amplifier noise. Work is now in progress to optimise F_{DAG} for other filter circuits, with a view to intercomparison; this has so far shown that the Sallen and Key circuit, in spite of its poor sensitivity characteristics, is one of the designer's best choices as far as noise and distortion-aggravation are concerned.

References

[1] A. B. Haase and L. T. Bruton, "Noise optimisation in RC-active filter sections," *IEE J. Electron, Circuits and Syst.*, vol. 1, no. 4, pp. 117-124, 1977.

[2] P. J. Billam, "Harmonic Distortion in a Class of Linear Active Filter Networks", *J. Audio Eng. Soc.*, vol. 26 no. 6, pp.426-429, June 1978

[3] R. P. Sallen and E. L. Key, "A practical method of designing RC-active filters," *IRE Trans. Circuit Theory*, vol. CT-2, pp74-85, 1955.

[4] W. Saraga, "Sensitivity of 2nd oder Sallen-Key type active RC filters," *Electron. Lett.*, pp. 442-444, Oct 1967.

11 AUTHOR'S COMMENT IN 2017

I seem to have had the unstated assumption that $\xi > 1$ and $\gamma > 1$
But I don't see a reason why they can't also be less than one.
So those Component Constraints $C/\gamma > C_{min}$, and $R/\xi > R_{min}$ should also have mentioned the conditions:

$$C\,/\,\gamma < C_{max} \text{ and } R\,/\,\xi < R_{max} \text{ and}$$
$$C \cdot \gamma > C_{min} \text{ and } R \cdot \xi > R_{min} \text{ and}$$
$$C \cdot \gamma < C_{max} \text{ and } R \cdot \xi < R_{max}$$

That makes eight constraints, so the multiplication $\xi\gamma$ which yields [6] and [7], which is only two constraints, is already suspect from an information-theoretic point of view.
E.g. [6] and [7] bypass all component-constraints if γ is extremely large and ξ extremely small, or vice versa.

Thus "since Q and $\xi\gamma$ remain constant" is wrong, and the rest of the paper falls apart.
Indeed, it falls apart starting with "It will shortly be seen that", just after Eqn. [5].

Concretely, step 2) of the "practical procedure"

2) $\xi\gamma$ is chosen not to contavene either of the conditions (6) or (7)

is undermined.

And Eqn. [12] is obviously doubtful, because of: $\sqrt{\xi\gamma/Q - 1}$
So what happens when $Q > \xi\gamma$ and the square root is imaginary ?
For example if $\xi = 2$ and $\gamma = 2$, then $Q = 5$ so that $(4/5 - 1)$ is negative.

In summary, if indeed $\xi > 1$ and $\gamma > 1$ does produce an optimisation, which it often does, then this paper works. Otherwise it doesn't work.

A more reliable source is the paper by Oscar Juan Bonello on "Distortion in Positive- and Negative-Feedback Filters", published in the *Journal of the Audio Engineering Society* in April 1984.
See: *http://www.pjb.com.au/comp/index.html#electronics*
Or: *http://www.pjb.com.au/comp/free/1984_JAES_bonello_paper.pdf*